Prop 1: Let X be a normed space. Then the following assertions are equivalent.

(i) X is a Banach space.

$\Updownarrow$

(ii) If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in X, i.e $\sum_{n=1}^{\infty} \|x_n\| < \infty$, implies that the series $\sum_{n=1}^{\infty} x_n$ converges in norm.

Pf:

(i) $\Rightarrow$ (ii)

Let $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in X and $S_n = \sum_{k=1}^{m} x_k$.

For any $\varepsilon > 0$, $(n>m)$

$$\| S_n - S_m \| \le \sum_{k=m+1}^{n} \|x_k\| < \varepsilon \quad , \text{ for } n, m \text{ large enough}.$$

$\{S_n\}$ is a cauchy sequence.

Since X is a Banach space,

$\exists\, S \in X$, $\quad \|S_n - S\| < \varepsilon$ for n large enough.

That is, $\| \sum_{k=1}^{n} x_k - S \| < \varepsilon$ for n large enough

(ii) $\Rightarrow$ (i)

Let $\{S_n\}_{n\ge 0}$ be a cauchy sequence with $S_0 = 0$.

Let $x_n = S_n - S_{n-1}$, $\quad \forall\, n \ge 1$.

For any $k > 0$, $\exists\, n_k \in \mathbb{N}$,

$$\| x_{n_k} \| = \| S_{n_k} - S_{n_k - 1} \| < \frac{1}{2^k}$$

$$\sum_{k=1}^{\infty} x_{n_k} \quad , \quad \sum_{k=1}^{\infty} \|x_{n_k}\| < \infty .$$

By iii) $\exists S \in X$, $\sum_{k=1}^{\infty} x_{n_k}$ converges to S in the norm.

$$\sum_{k=1}^{m} x_{n_k} = (x_{n_1} + x_{n_2} + \cdots \; x_{n_m}$$
$$= S_{n_1} - S_{n_{1-0}} + (S_{n_2} - S_{n_1}) + \cdots + (S_{n_m} - S_{n_{m-1}})$$
$$= S_{n_m}$$

$$\|S_{n_m} - S\| = \left\| \sum_{k=1}^{m} x_{n_k} - S \right\| < \varepsilon .$$

Recall the definition of dual space.

- Let X be a normed space. The set of bounded linear fcts on X constitutes a normed space with norm defined by
$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| .$$
which is called the dual space of X and denoted by X^*.
- An isomorphism of a normed space X onto a normed space Y is bijective linear operator $T: X \to Y$ which preserves the norm.
$$\|TX\|_Y = \|x\|_X .$$

Thm : The dual space of l^p is the space l^q, where p, q are Hölder' conjugates. i.e

$$(l^p)^* \cong l^q, \quad 1 < p < \infty$$

Pf:

$$(l^p)^* \subset l^q$$

$$(T : (l^p)^* \longrightarrow l^q, \text{ st } T((l^p)^*) \subset l^q).$$

Construct an injective operator T.

Such that $\underbrace{\|Tf\|_{l^q} \le \|f\|}_{\in l^q}, < \infty$ where $f \in (l^p)^*$ $(\Rightarrow \|T\| \le 1)$

For any $x \in l^p$, there exists a unique sequence of $x_k \in \mathbb{R}$, st $x = \sum x_k e_k$, where e_k is the Schauder basis of l^p.

Then $f(x) = \sum_{k=1}^{\infty} x_k f(e_k)$, since f is linear.

Denote $f(e_k)$ by b_k and we can define an injective operator T by

$$T(f) = (b_k) = (f(e_k))$$

- T is injective.

  If $f, g \in (l^p)^*$ with $f \neq g$. Then $\exists\, e_k$, st $f(e_k) \neq g(e_k)$

  $\Rightarrow Tf \neq Tg$.

Construct a sequence $x^n = (x^n_k)$ as

$$x^n_k = \begin{cases} \dfrac{|b_k|^q}{b_k}, & \text{if } b_k \neq 0 \text{ and } k \le n \\ 0, & \text{otherwise} \end{cases}$$

Then $x^n = (x^n_k) \in l^p$. . . . .

and $f(x^n) = \sum x_k^n f(e_k) = \sum x_k^n b_k$

$$= \sum_{k=1}^{n} \frac{|b_k|^q}{b_k} \cdot b_k = \sum_{k=1}^{n} |b_k|^q$$

By the boundedness of f,

$$\underline{\sum_{k=1}^{n} |b_k|^q} = |f(x^n)| \le \|f\| \cdot \|x^n\|_{l^p}$$
$$= \|f\| \left(\sum_{k=1}^{n} |x_k^n|^p \right)^{\frac{1}{p}}$$
$$= \|f\| \left(\sum_{k=1}^{n} |b_k|^{(q-1)p} \right)^{\frac{1}{p}}$$
$$= \underline{\|f\| \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{p}}}$$

Therefore, $\left(\sum_{k=1}^{n} |b_k|^q \right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}} \le \|f\|$

let $n \to \infty$ then

$$\left(\sum_{k=1}^{\infty} |b_k|^q \right)^{\frac{1}{q}} = \|(b_k)\|_{l^q} = \|Tf\|_{l^q} \le \|f\|$$

Step 2: $l^q \subset (l^p)^*$ $\quad \|Tf\|_{l^q} = \|f\|$, surjective

(For any $(b_k) \in l^q$, there exists $f \in (l^p)^*$, st $Tf = (b_k)$

For an arbitrary sequence $(b_k) \in l^q$.

it can be checked that the mapping

$$f : l^p \to \mathbb{R}$$

$$f(x) := \sum_{k=1}^{\infty} x_k b_k, \quad \forall\, x = (x_k) \in l^p.$$

$$|f(x)| \leq \sum_{k=1}^{\infty} |x_k b_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum |b_k|^q\right)^{\frac{1}{q}} < \infty$$

$$= \|x\|_{l^p} \cdot \|(b_k)\|_{l^q}$$

$f \in (l^p)^*$

By construction of f, we have

$f(e_k) = b_k$, which implies that T is surjective,

and $\|f\| \leq \|(b_k)\|_{l^q} = \|Tf\|_{l^q}$

$\Rightarrow \|T\| \geq 1$.

Thm. The dual space l^1 is the space l^∞.